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NOTE

The Instability of the Yee Scheme for the "Magic Time Step"

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1. INTRODUCTION

The Yee algorithm [1] for the Maxwell equations requires that the time step Δt be bounded in order to avoid numerical instability. In [2], it was stated that the scheme is stable under the conditions $\Delta t \leq \Delta x$ in one dimension and $\Delta t \leq 1/\sqrt{(1/\Delta x)^2 + (1/\Delta y)^2}$ in two dimensions.

In this paper, we show that the Yee scheme is not stable when $\Delta t = \Delta x$ for the one-dimensional case and $\Delta t = 1/\sqrt{(1/\Delta x)^2 + (1/\Delta y)^2}$ for the two-dimensional case. This means that one cannot take the maximum Δt referred to as the "magic time step" in [2]. Remis [5] found similar results by studying the eigenvalues of the iteration matrix for a specific boundary condition. However, in this paper, the analysis is carried out using the Kreiss matrix theorem [4] for the case with a periodic boundary condition.

We consider the fully discrete Yee scheme applied to the dimensionless form of Maxwell's equations in free space. Fourier transformation of the scheme gives us a linear system with an amplification matrix $\hat{Q}$. In order to analyze the stability of the scheme, we investigate the properties of the amplification matrix.

In Section 2, we first examine the stability of the one-dimensional scheme. It is shown that the scheme with the condition $\Delta t > \Delta x$ does not satisfy the von Neumann condition [4], which implies that one of the eigenvalues of the amplification matrix is greater than one in magnitude and so the scheme is unstable. For the case $\Delta t < \Delta x$, the von Neumann condition becomes a necessary and sufficient condition for the stability due to the fact that

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the norm of the nth power of $\hat{Q}$ is uniformly bounded as n grows. Finally, we analyze the case $\Delta t = \Delta x$, which is our main concern in this paper. This satisfies the von Neumann condition. However, the norm of the nth power of $\hat{Q}$ grows linearly with n, so that the numerical scheme is unstable for this case.

In Section 3, we extend the same stability analysis to the two-dimensional scheme. A detailed proof is shown only for the case $\Delta t = 1/\sqrt{(1/\Delta x)^2 + (1/\Delta y)^2}$.

We provide a numerical example of the instability of the magic time step for a one-dimensional problem in Section 4 and the conclusion is in the last section.

2. THE ONE-DIMENSIONAL CASE

The non-dimensional Maxwell equations, describing the dynamics of waves in one-dimensional free space, are written as

$$\begin{aligned} \frac{\partial H_y}{\partial t} &= \frac{\partial E_z}{\partial x} \\ \frac{\partial E_z}{\partial t} &= \frac{\partial H_y}{\partial x}, \end{aligned} \tag{1}$$

where H_y and E_z are the magnetic field in the y direction and the electric field in the z direction, respectively.

The Yee scheme applied to the above equations is

$$\begin{aligned} H_y\big|_{j+\frac{1}{2}}^{n+\frac{3}{2}} - H_y\big|_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= \frac{\Delta t}{\Delta x}\left(E_z\big|_{j+1}^{n+1} - E_z\big|_{j}^{n+1}\right) \\ E_z\big|_{j}^{n+1} - E_z\big|_{j}^{n} &= \frac{\Delta t}{\Delta x}\left(H_y\big|_{j+\frac{1}{2}}^{n+\frac{1}{2}} - H_y\big|_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right). \end{aligned} \tag{2}$$

Consider the sinusoidal-traveling-wave solution of (1) as numerically evaluated at the discrete space–time point (x_j, t_n),

$$\begin{aligned} H_y\big|_j^n &= H_y(j\Delta x, n\Delta t) = H_y(x_j, t_n) = \hat{H}_y^n(w)e^{iwx_j} \\ E_z\big|_j^n &= E_z(j\Delta x, n\Delta t) = E_z(x_j, t_n) = \hat{E}_z^n(w)e^{iwx_j}, \end{aligned} \tag{3}$$

where w is the wavenumber.

Substituting (3) into the Yee scheme (2) and canceling the common terms, we obtain the system

$$\begin{bmatrix} \hat{H}_y^{n+\frac{3}{2}} \\ \hat{E}_z^{n+1} \end{bmatrix} = \begin{bmatrix} 1 - 4\lambda^2 \sin^2\frac{\xi}{2} & 2i\lambda\sin\frac{\xi}{2} \\ 2i\lambda\sin\frac{\xi}{2} & 1 \end{bmatrix} \begin{bmatrix} \hat{H}_y^{n+\frac{1}{2}} \\ \hat{E}_z^{n} \end{bmatrix}, \tag{4}$$

where $\lambda = \Delta t/\Delta x$ and $\xi = w\Delta x$. Thus the amplification matrix $\hat{Q}$ is given by

$$\begin{bmatrix} 1 - 4\alpha^2 & 2i\alpha \\ 2i\alpha & 1 \end{bmatrix} \tag{5}$$

with $\alpha = \lambda \sin \frac{\xi}{2}$. The corresponding eigenvalues of the matrix $\hat{Q}$ are

$$\mu_{1,2} = 1 - 2\alpha^2 \pm 2\sqrt{\alpha^4 - \alpha^2}. \tag{6}$$

A necessary condition for the stability is that all the eigenvalues of the amplification matrix $\hat{Q}$ must be less than or equal to one in magnitude (the von Neumann condition [4]).

First, assume that $\lambda > 1$. By letting $\sin(\xi/2) = 1$, we have $\alpha^2 > 1$. Then one of the eigenvalues of $\hat{Q}$ is greater than one in magnitude; i.e.,

$$|\mu_2| = |1 - 2\alpha^2 - 2\sqrt{\alpha^4 - \alpha^2}| > 1. \tag{7}$$

Thus the scheme is unstable for $\lambda > 1$.

Next, assume that

$$0 < \lambda \leq 1.$$

Then we have

$$|\mu_{1,2}|^2 = (1 - 2\alpha^2)^2 + 4\alpha^2(1 - \alpha^2) = 1, \tag{8}$$

so that the von Neumann condition is satisfied. In general, the von Neumann condition is only a necessary condition but not a suffcient condition. So it does not guarantee that the scheme is stable for $0 < \lambda \leq 1$. However, in the special case that $\hat{Q}$ can be uniformly diagonalized, the von Neumann condition is sufficient [4].

Here we show that the scheme is stable for $0 < \lambda < 1$, but not for the case $\lambda = 1$. We separately investigate the two different cases.

1. Suppose $0 < \lambda < 1$. Define the diagonalizer T by

$$T = \begin{bmatrix} \beta + i\alpha & -\beta + i\alpha \\ 1 & 1 \end{bmatrix}$$

such that

$$T^{-1} = \frac{1}{2\beta} \begin{bmatrix} 1 & \beta - i\alpha \\ -1 & \beta + i\alpha \end{bmatrix}$$

and

$$T^{-1}\hat{Q}T = \Lambda = \mathrm{diag}(\mu_1, \mu_2),$$

where $\beta = \sqrt{\alpha^2 - \alpha^4}/\alpha$. Let T^* be the hermitian matrix of T and $\rho(T^* \cdot T)$ be the spectral radius of $T^* \cdot T$. Then the norms of T and T^{-1} are bounded as the following:

$$\begin{aligned} \|T\|_2^2 &= \rho(T^* \cdot T) = 2 + 2|\alpha| < 4, \\ \|T^{-1}\|_2^2 &= \rho(T^{-1^*} \cdot T^{-1}) = \frac{1 + \sqrt{\alpha^2}}{2(1 - \alpha^2)} = \frac{1}{2(1 - |\alpha|)}. \end{aligned}$$

Thus

$$\begin{aligned} \|\hat{Q}^n\|_2 &= \|T\Lambda^n T^{-1}\|_2 \\ &\leq \|T\|_2 \cdot \|\Lambda^n\|_2 \cdot \|T^{-1}\|_2 \\ &\leq C \end{aligned} \tag{9}$$

for some constant C. Therefore, the Yee scheme is stable for the case $0 < \lambda < 1$.

2. Suppose $\lambda = 1$. We show that, in this case, the Yee scheme is unstable. Let $\xi = \pi$. Then the amplification matrix can be written as

$$\hat{Q} = \begin{bmatrix} -3 & 2i \\ 2i & 1 \end{bmatrix} = -I + \begin{bmatrix} -2 & 2i \\ 2i & 2 \end{bmatrix} = -I + B. \tag{10}$$

Since $B^n = 0$ for $n \geq 2$, we have

$$\begin{aligned} \hat{Q}^n &= \sum_{m=0}^{n} \binom{n}{m} B^m (-I)^{n-m} \\ &= (-1)^n I + nB(-1)^{n-1} I \\ &= (-1)^n (I - nB) \\ &= (-1)^n \begin{bmatrix} 1+2n & -2ni \\ -2ni & 1-2n \end{bmatrix}. \end{aligned} \tag{11}$$

Each of the entries of the matrix $\hat{Q}^n$ grows linearly with n. Therefore, the norm of $\hat{Q}^n$ is of order n, which cannot be uniformly bounded.

3. THE TWO-DIMENSIONAL CASE

Here, we examine the stability criteria of the Yee scheme in two dimensions. The dimensionless form of Maxwell's equations in two-dimensional free space is

$$\begin{aligned} \frac{\partial H_x}{\partial t} &= -\frac{\partial E_z}{\partial y} \\ \frac{\partial H_y}{\partial t} &= \frac{\partial E_z}{\partial x} \\ \frac{\partial E_z}{\partial t} &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}, \end{aligned} \tag{12}$$

where H_x, H_y, and E_z are the field components in the x, y, and z directions.

Applying the Yee scheme to the equations above, we have

$$\begin{aligned} H_x\Big|_{j,k+\frac{1}{2}}^{n+\frac{3}{2}} - H_x\Big|_{j,k+\frac{1}{2}}^{n+\frac{1}{2}} &= -\frac{\Delta t}{\Delta y}\left(E_z\Big|_{j,k+1}^{n+1} - E_z\Big|_{j,k}^{n+1}\right) \\ H_y\Big|_{j+\frac{1}{2},k}^{n+\frac{3}{2}} - H_y\Big|_{j+\frac{1}{2},k}^{n+\frac{1}{2}} &= \frac{\Delta t}{\Delta x}\left(E_z\Big|_{j+1,k}^{n+1} - E_z\Big|_{j,k}^{n+1}\right) \\ E_z\Big|_{j,k}^{n+1} - E_z\Big|_{j,k}^{n} &= \frac{\Delta t}{\Delta x}\left(H_y\Big|_{j+\frac{1}{2},k}^{n+\frac{1}{2}} - H_y\Big|_{j-\frac{1}{2},k}^{n+\frac{1}{2}}\right) \\ &\quad - \frac{\Delta t}{\Delta y}\left(H_x\Big|_{j,k+\frac{1}{2}}^{n+\frac{1}{2}} - H_x\Big|_{j,k-\frac{1}{2}}^{n+\frac{1}{2}}\right). \end{aligned} \tag{13}$$

Consider a simple wave solution of (12) at the discrete space–time point $(x_j,\ y_k,\ t_n)$,

$$\begin{aligned} H_x|_{j,k}^n &= H_x(j\Delta x, k\Delta y, n\Delta t) = H_x(x_j, y_k, t_n) = \hat{H}_x^n(w_x, w_y)e^{i(w_x x_j + w_y y_k)} \\ H_y|_{j,k}^n &= H_y(j\Delta x, k\Delta y, n\Delta t) = H_y(x_j, y_k, t_n) = \hat{H}_y^n(w_x, w_y)e^{i(w_x x_j + w_y y_k)} \\ E_z|_{j,k}^n &= E_z(j\Delta x, k\Delta y, n\Delta t) = E_z(x_j, y_k, t_n) = \hat{E}_z^n(w_x, w_y)e^{i(w_x x_j + w_y y_k)}. \end{aligned} \tag{14}$$

Substituting the solution (14) into the Yee scheme (13) gives us the system

$$\begin{bmatrix} \hat{H}_x^{n+\frac{3}{2}} \\ \hat{H}_y^{n+\frac{3}{2}} \\ \hat{E}_z^{n+1} \end{bmatrix} = \begin{bmatrix} 1-4\alpha_y^2 & 4\alpha_x\alpha_y & -2i\alpha_y \\ 4\alpha_x\alpha_y & 1-4\alpha_x^2 & 2i\alpha_x \\ -2i\alpha_y & 2i\alpha_x & 1 \end{bmatrix} \begin{bmatrix} \hat{H}_x^{n+\frac{1}{2}} \\ \hat{H}_y^{n+\frac{1}{2}} \\ \hat{E}_z^{n} \end{bmatrix}, \tag{15}$$

where

$$\alpha_x = \lambda_x \sin\frac{\xi_x}{2}, \quad \alpha_y = \lambda_y \sin\frac{\xi_y}{2}$$

with

$$\lambda_x = \Delta t/\Delta x, \quad \lambda_y = \Delta t/\Delta y \qquad \text{and} \qquad \xi_x = w_x\Delta x, \quad \xi_y = w_y\Delta y.$$

Then the amplification matrix $\hat{Q}$ is

$$\begin{bmatrix} 1-4\alpha_y^2 & 4\alpha_x\alpha_y & -2i\alpha_y \\ 4\alpha_x\alpha_y & 1-4\alpha_x^2 & 2i\alpha_x \\ -2i\alpha_y & 2i\alpha_x & 1 \end{bmatrix}. \tag{16}$$

Defining $\alpha = \alpha_x^2 + \alpha_y^2$, the eigenvalues of $\hat{Q}$ are

$$\mu_1 = 1, \quad \mu_{2,3} = 1 - 2\alpha \pm 2\sqrt{\alpha^2 - \alpha}. \tag{17}$$

One can easily show that the scheme is unstable for $\lambda_x^2 + \lambda_y^2 > 1$ using the same analysis as in the one-dimensional case. On the other hand, the scheme is stable when $0 < \lambda_x^2 + \lambda_y^2 < 1$. The proof is omitted.

Here, we will only show a detailed proof for the case $\lambda_x^2 + \lambda_y^2 = 1$. With $\xi_x = \xi_y = \pi$ and $\lambda_x = \lambda_y = 1/\sqrt{2}$, the amplification matrix becomes

$$\hat{Q} = \begin{bmatrix} -1 & 2 & -i\sqrt{2} \\ 2 & -1 & i\sqrt{2} \\ -i\sqrt{2} & i\sqrt{2} & 1 \end{bmatrix}.$$

Let P be defined as

$$P = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -i\sqrt{2} & 0 \end{bmatrix}$$

so that $\hat{Q}$ is transformed to the Jordan canonical form

$$P^{-1}\hat{Q}P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then

$$\hat{Q}^n = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & (-1)^{n-1}n \\ 0 & 0 & (-1)^n \end{bmatrix} P^{-1}.$$

Here, $\|\hat{Q}^n\|$ is unbounded when $n \to \infty$. Therefore, the scheme is unstable under the condition $\lambda_x^2 + \lambda_y^2 = 1$; i.e., $\Delta t = 1/\sqrt{(1/\Delta x)^2 + (1/\Delta y)^2}$.

4. NUMERICAL RESULTS

We provide a numerical example demonstrating the instability of the numerical scheme with the condition $\Delta t = \Delta x$ in one dimension. Consider the exact solution of Maxwell's equations (1) given by

$$E(x, t) = H(x, t) = \sin(x + t).$$

We choose two different sets of grids such that, for even integer N,

$$x_i = \frac{2\pi i}{N+1}, \qquad i = 0, 1, 2, \ldots, N, \tag{18}$$

and

$$x_i = \frac{2\pi i}{N}, \qquad i = 0, 1, 2, \ldots, N-1. \tag{19}$$

We then apply the Yee scheme with the magic time step $\Delta t = \Delta x$ and the initial data

$$E_i^0 = \sin(x_i) + \delta \cos\left(\frac{N}{2} x_i\right) \tag{20}$$

$$H_{i+1/2}^{1/2} = \sin\left(x_{i+1/2} + 0.5\Delta t\right). \tag{21}$$

Here we introduce the perturbed initial data E^0 for the electric field E by adding the term $\delta \cos(\frac{N}{2} x_i)$ with $\delta \ll 1$. On the other hand, the exact solution is considered initial data for the magnetic field H such as $H^{1/2}$ in (21).

We measure the error between the numerical solution E_i^n and the exact solution E_* of the electric field in the L_2-norm defined by

$$\varepsilon = \sqrt{\Delta x \cdot \sum_i \left|E_i^n - E_*(x_i, t_n)\right|^2}.$$

In Table I, the errors on the two different sets of grids at time $t = 20\pi$ are compared. With the grids $x_i = \frac{2\pi i}{N}$, $i = 0, 1, \ldots, N-1$, the error grows linearly with N for fixed terminal time. Since $\hat{E}^0(\omega = \pm N/2) \neq 0$ due to the perturbation term $\delta \cos(\frac{N}{2} x_i)$ and the variable $\xi = \omega \Delta x = \pi$, we have $\hat{E}^{n+1}(\omega = \pm N/2)$ being amplified by $\hat{Q}^n$ as shown in (11). On the other hand, taking the grids $x_i = \frac{2\pi i}{(N+1)}$, $i = 0, 1, \ldots, N$, we have $\xi = w\Delta x = \frac{N}{2} \cdot \frac{2\pi}{N+1} < \pi$, so that the instability is not expected for any finite N for this case.

TABLE I

L_2 Error of E for Various δ's on Various Sets of Grids at Time $t = 20\pi$

	$x_i = \frac{2\pi i}{(N+1)}, i = 0, 1, \ldots, N$			$x_i = \frac{2\pi i}{N}, i = 0, 1, \ldots, N-1$		
N	$\delta = 1.E\text{-}3$	$\delta = 1.E\text{-}9$	$\delta = 1.E\text{-}15$	$\delta = 1.E\text{-}3$	$\delta = 1.E\text{-}9$	$\delta = 1.E\text{-}15$
8	0.4896E-02	0.4896E-08	0.1288E-13	0.3985E+00	0.3985E-06	0.4446E-12
16	0.4885E-02	0.4885E-08	0.8921E-14	0.7996E+00	0.7996E-06	0.7549E-12
32	0.4881E-02	0.4881E-08	0.8826E-14	0.1601E+01	0.1601E-05	0.1554E-11
64	0.4880E-02	0.4880E-08	0.9609E-14	0.3206E+01	0.3206E-05	0.3028E-11
128	0.4879E-02	0.4879E-08	0.1016E-13	0.6414E+01	0.6414E-05	0.6151E-11
256	0.4879E-02	0.4879E-08	0.1034E-13	0.1283E+02	0.1283E-04	0.1234E-10
512	0.4879E-02	0.4879E-08	0.1037E-13	0.2566E+02	0.2566E-04	0.2465E-10
1024	0.4879E-02	0.4879E-08	0.1025E-13	0.5133E+02	0.5133E-04	0.4898E-10
2048	0.4879E-02	0.4879E-08	0.1031E-13	0.1026E+03	0.1026E-03	0.9744E-10
4096	0.4879E-02	0.4879E-08	0.1043E-13	0.2053E+03	0.2053E-03	0.1967E-09

5. CONCLUSION

We have proven that the Yee scheme with the magic time step is not stable. In a real computation, one always expects perturbations from either measurement errors in the data or roundoff errors. From the numerical results for the one-dimensional problem, the linear instability for the case of the magic time step is detected when a small perturbation is introduced in such a way as shown in this paper. Therefore, we conclude that the magic time step is not suitable for the Yee scheme since the solution may diverge under a certain small perturbation.

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